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<u>spring action (harmonic motion)</u>

$mx'' + cx' + kx = f(t)$

let $y_1 = x$, $y_2 = x'$, $y_1(0) = x_0$, $y_2(0) = v_0$

$\therefore y_1' = y_2$

$y_2' = -\frac{k}{m} y_1 - \frac{c}{m} y_2 + \frac{f(t)}{m}$

$$\therefore \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{f(t)}{m} \end{pmatrix}$$

$$\underline{\underline{Y' = AY + B}}$$

set ~~var~~ $f(t) = 0$,

$Y' = AY$

let $Y = \begin{pmatrix} a \\ b \end{pmatrix} e^{\lambda t}$

$\therefore \lambda \begin{pmatrix} a \\ b \end{pmatrix} - A \begin{pmatrix} a \\ b \end{pmatrix} = 0$

$\det[\lambda I - A] = 0$

$\therefore \lambda(\lambda + \frac{c}{m}) + \frac{k}{m} = 0$

$$\lambda = \frac{-\frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}}}{2} = \lambda_1, \lambda_2$$

$\therefore Y(t) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} e^{\lambda_1 t} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} e^{\lambda_2 t}$

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initial condition $(t=0)$:

ξ $\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}$ $\because y_1' = y_2$

$\therefore b_1 + b_2 = a_1\lambda_1 + a_2\lambda_2$

First-order Systems

$$X'(t) = A\,X(t), \quad X(\tau) = \xi$$

$A : I \to M_{nn}(F)$, F is a field, I is real interval,

$n \times n$ matrix $\quad \xi \in F^n$

solution set S: X is a vector space over F

$X_1, X_2, \cdots, X_k$ linearly independent $\Leftrightarrow$

$X_1(\tau) = F_1, X_2(\tau) = F_2, \cdots X_k(\tau) = F_k$ linearly independent in F^n

if $k = n$, they form basis.

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

$$= \tilde{X} C$$

$$= (X_1\ X_2 \cdots X_n)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

if $\hat{X}$ invertible, if $\hat{X}C = 0$

$C = \hat{X}^{-1}\hat{X}C = 0 \Rightarrow$ independent

if $\hat{X}$ independent, if $\hat{X}C = 0$, $t = \tau$

$\therefore X = \hat{X}C = 0$, since $\hat{X}$ is basis,

$\therefore C = 0$, $\Rightarrow \hat{X}$ invertible

$\because$ null vector $= 0$

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if $\tilde{X}(\tau) = I_n$

$X = \tilde{X} C$, $\because \tilde{X}(\tau) C = I_n C$

The Wronskian

$$W_X(t) = \det[\tilde{X}(t)], \quad X = (X_1, X_2, \cdots, X_n)$$

$$W_X(t) = \sum \varepsilon x_1 x_2 \cdots x_N$$

$$W_X'(t) = \sum \varepsilon (x_1' x_2 \cdots x_N + x_1 x_2' \cdots x_N + \cdots + x_1 x_2 \cdots x_N')$$

$$= \det\begin{pmatrix} x_{11}' & x_{12}' & \cdots & x_{1n}' \\ & \ddots & & \\ & & & x_{nn} \end{pmatrix} + \det\begin{pmatrix} x_{11} & x_{12} & \cdots \\ x_{21}' & x_{22}' & \cdots \\ & & \ddots & x_{nn} \end{pmatrix}$$

$$+ \cdots + \det\begin{pmatrix} x_{11} & \cdots & \\ \vdots & \ddots & \\ x_{n1}' & x_{n2}' & x_{nn}' \end{pmatrix}$$

$$= w_1 + \cdots + w_n$$

$\because \tilde{X}'(t) = A\tilde{X}(t)$ $\quad\therefore w_1 = \det\begin{pmatrix} \sum a_{1j} x_{j1} & \sum a_{1j} x_{j2} & \cdots \\ x_{21} & \ddots & \\ \vdots & & x_{nn} \end{pmatrix}$

$x_{11}' = \sum_j a_{1j} x_{j1}$

$x_{12}' = \sum_j a_{1j} x_{j2}$

$$= \det\begin{pmatrix} a_{11} x_{11} & a_{11} x_{12} & \cdots \\ x_{21} & \ddots & \\ \vdots & & x_{nn} \end{pmatrix} = a_{11} W_X(t)$$

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$$\therefore W_X'(t) = tr(A) W_X(t)$$

$$W_X(t) = W_X(\tau) e^{\int_\tau^t tr(A)(s)\,ds}, \quad t, \tau \in I$$

First-order nonhomogeneous systems

$$X'(t) = AX(t) + B(t)$$

let $X_p(t)$ be the particular solution, $U(t)$ be the homogeneous solution, i.e. $U'(t) = AU(t)$, because

~~XXXX~~ $U(t) = X(t) - X_p(t)$ where $X(t)$ is any other solution to the nonhomogeneous systems

$$U'(t) = X'(t) - X_p'(t)$$
$$= A(X(t) - X_p(t))$$
$$= AU(t)$$

conversely, $X(t) = X_p(t) + U(t)$

$$X'(t) = X_p'(t) + U'(t)$$
$$= AX_p(t) + B(t) + AU(t)$$
$$= AX(t) + B(t)$$

$$\therefore X(t) = \underbrace{X_p(t)}_{\text{particular}} + U(t)$$

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Examples:

$x_1' = x_2$

$x_2' = -x_1$ $\qquad t \in [0, 9]$

$$X' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = AX \qquad X(0) = \xi$$

$$U(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \qquad V(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

since $U(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, they form the basis of solution space.

$$\therefore X = \begin{pmatrix} U(t) & V(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$x_1' = x_2 + 1$

$x_2' = -x_1 + 2$

$$\tilde{X}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \qquad X_p(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore X(t) = X_p(t) + \tilde{X} C = \begin{pmatrix} 2 + c_1 \cos t + c_2 \sin t \\ 1 - c_1 \sin t + c_2 \cos t \end{pmatrix}$$

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variation of parameters

$X(t) = \tilde{X}(t) C(t)$, solution of homogeneous system,

$$X'(t) = \tilde{X}'(t) C(t) + \tilde{X}(t) C'(t)$$
$$= A\tilde{X}(t) C(t) + \tilde{X}(t) C'(t)$$
$$= AX(t) + \tilde{X}(t) C'(t) \overset{!}{=} AX(t) + B(t)$$

$$\tilde{X}(t)\,C'(t) = B(t)$$

$$\therefore\ C'(t) = \tilde{X}^{-1}(t)\,B(t)$$

$$\therefore\ C(t) = \xi + \int_\tau^t \tilde{X}^{-1}(t)\,B(t)\,dt$$

$$X(t) = \tilde{X}(t)\,\xi + \tilde{X}(t)\int_\tau^t \tilde{X}^{-1}(s)\,B(s)\,ds$$

if $\tilde{X}(\tau) = I_n$, then $X(\tau) = \xi$

Examples:

$$\tilde{X}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \tilde{X}^{-1}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$\therefore\ X(t) = \tilde{X}(t)\,\xi + \tilde{X}(t)\int_\tau^t \begin{pmatrix} \cos t - 2\sin t \\ \sin t + 2\cos t \end{pmatrix} dt, \quad X(0) = 0,\ \tau = 0$$

$$= \tilde{X}(t)\,\xi + \tilde{X}(t)\begin{pmatrix} \sin t + 2\cos t \cancel{\xi} - 2 \\ -\cos t + 2\sin t \cancel{\xi} + 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \sin t + 2\cos(t) - 2 \\ -\cos t + 2\sin(t) + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 - 2\cos t + \sin t \\ -1 + 2\sin t + \cos t \end{pmatrix}$$

Linear equations of order n

$$x^{(n)} + a_{n-1}x^{(n-1)} + \dots + a_0 x = b(t)$$

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$$y_1 = x$$
$$y_2 = x'$$
$$\vdots$$
$$y_n = x^{(n-1)} \quad \Rightarrow \quad y_n' = -a_0 y_1 - a_1 y_2 - \cdots - a_{n-1} y_n + b$$

$$\therefore Y(t) = A(t) Y(t) + B(t), \quad Y(t) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$A(t) = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \vdots \\ & & & \ddots & \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{pmatrix} \quad B(t) = \begin{pmatrix} 0 \\ \vdots \\ b \end{pmatrix}$$

let $\tilde{x} = \begin{pmatrix} x \\ x^{(1)} \\ \vdots \\ x^{(n-1)} \end{pmatrix}$

$x_1(t), \dots, x_k(t)$ linearly independent $\Leftrightarrow$
$\tilde{x}_1(\tau), \dots, \tilde{x}_k(\tau)$ linearly independent

$$W_x = \det \begin{pmatrix} x_1 & & x_n \\ x_1^{(1)} & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ x_1^{(n-1)} & & x_n^{(n-1)} \end{pmatrix}$$

$$W_x(t) = W_x(\tau)\, e^{\int_\tau^t -a_{n-1}(s)\, ds}$$

<u>Nonhomogeneous linear equations</u>

according to the variation of parameters method;

$$Y(t) = \tilde{X}(t)C(t)$$

$$\tilde{X}(t)C'(t) = B(t)$$

$$\therefore \tilde{X}C' = bE_n \,, \quad E_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$C(t) = \int_\tau^t \tilde{X}^{-1}(s)b(s)E_n \, ds$$

$$x_p(t) = \tilde{X}(t)C(t) = X(t)\int_\tau^t \tilde{X}^{-1}(s)\,b(s)E_n \, ds \,, \quad X(t) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

if coeff. of $x^{(n)} \neq 1$, let's say a_n,

$$x_p(t) = X(t)\int_\tau^t \tilde{X}^{-1}(s)\frac{b(s)}{a_n(s)}E_n \, ds$$

<u>Examples:</u>

$$x^{(3)} - x^{(1)} = t$$

let $x^{(3)} - x^{(1)} = 0$, $\quad x = 1, e^t, e^{-t}$

$$\tilde{X}(t) = \begin{pmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{pmatrix}, \quad W_X(t) = 2$$

$$c_1' + e^t c_2' + e^{-t} c_3' = 0$$

$\tilde{X}(t)C'(t) = B(t)$ $\quad e^t c_2' - e^{-t} c_3' = 0 \quad \Rightarrow c_2' \sim e^{-t}, \; c_3' \sim e^t$

$\quad e^t c_2' + e^{-t} c_3' = t \quad \Rightarrow c_2' \sim (a+bt)e^{-t}$

$\quad c_3' \sim (a+bt)e^{t}$

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$$C_1' = -\frac{t^2}{2}, \quad C_2' = -\frac{(1+t)}{2}e^{-t}, \quad C_3' = \frac{(t-1)}{2}e^{t}$$

$$\therefore x_p(t) = C_1(t) + e^t \cancel{\quad} C_2(t) + e^{-t} C_3(t)$$

$$= -\frac{t^2}{2} - 1$$

$$\therefore x(t) = -\frac{t^2}{2} + a_1 + a_2 e^t + a_3 e^{-t}$$

Constant coefficients

$$X' = AX$$

if A is diagonal, $A = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$

$$X(t) = \begin{pmatrix} e^{a_{11}t} & & & 0 \\ & e^{a_{22}t} & & \\ & & \ddots & \\ 0 & & & e^{a_{nn}t} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \quad X(0) = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

if A is diagonalizable, $A = QDQ^{-1}$, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$$(Q^{-1}X)' = D(Q^{-1}X)$$

$$Y' = DY$$

let $X(t) = C_0 + C_1 t + \cdots + C_n t^n + \cdots$

$$X'(t) = C_1 + 2C_2 t + 3C_3 t^2 + \cdots$$

$$AX(t) = AC_0 + AC_1 t + \cdots$$

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$X' = AX$ iff $(k+1)C_{k+1} = AC_k$

if $X(0) = \xi = C_0$, $C_k = \frac{A^k}{k!}\xi$

$\therefore X(t) = \xi + At\xi + \frac{1}{2!}A^2t^2\xi + \cdots = \sum_k \frac{A^k t^k}{k!}\xi = e^{At}\xi$

<u>Exponential of a matrix</u> :

$X' = AX$, $X(0) = I_n$, $X_A(t) = e^{At}$

let $Y(t) = X(s+t)$, $Y'(t) = X'(t+s) = AX(t+s) = AY(t)$

let $Z(t) = X(t)X(s)$, $Z'(t) = X'(t)X(s) = AX(t)X(s) = AZ(t)$

$Y(0) = Z(0) = X(s)$, $\therefore Y(t) = Z(t)$

$\therefore e^{A(s+t)} = e^{As}e^{At}$

$\Downarrow$

$I_n = e^{At}e^{-At} = e^{-At}e^{At} \Rightarrow X_A(t)X_A(-t) = I_n$

$X_A(t)X_A^{-1}(t) = I_n$

$|X_A(t)| = |X_A(0)|\, e^{\int_0^t tr(A)ds}$

$\therefore |e^{At}| = e^{tr(A)t}$

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if $AB = BA$, $X_A(t) = e^{At}$, $X_A'(t) = AX_A(t)$

$(BX_A(t))' = BX_A'(t) = BAX_A(t) = A(BX_A(t))$

$\therefore BX_A(t) = X_A(t)C$ [??]

$t=0 \Rightarrow B = C$, $\therefore Be^{At} = e^{At}B$

or $A^kB = BA^k \Rightarrow Be^{At} = e^{At}B$ similarly for $Ae^{Bt} = e^{Bt}A$

if $e^{At}B = Be^{At}$, let $Y = X_AX_B$

$Y' = X_A'X_B + X_AX_B' = AX_AX_B + X_ABX_B$

$= (A+B)X_AX_B = (A+B)Y$

$\therefore X_{A+B}' = (A+B)X_{A+B}$, $X_{A+B}(0) = I_n = Y(0)$

$\therefore X_{A+B} = Y = X_AX_B = X_BX_A$

$X_{A+B}'' = (A+B)^2X_{A+B}$

$X_{A+B}''(0) = (A+B)^2 = A^2 + AB + BA + B^2 = (X_AX_B)''(0)$

$= A^2 + 2AB + B^2$

$\therefore AB = BA$

$(\tilde{X} \sim X_A)$

$X' = AX + B(t)$, $\tilde{X}(\tau) = I_n$, $\tilde{X}(t) = e^{A(t-\tau)}$, $X(\tau) = \xi$

$$X(t) = \tilde{X}(t)\xi + \tilde{X}(t)\int_\tau^t X^{-1}(s)B(s)\,ds$$
$$= e^{A(t-\tau)}\xi + \int_\tau^t e^{A(t-\tau)} e^{-A(s-\tau)} B(s)\,ds$$
$$= e^{A(t-\tau)}\xi + \int_\tau^t e^{A(t-s)} B(s)\,ds$$

Solution space of constant coefficients

define $L(X) = X' - AX$

let $X = e^{\lambda t}\alpha$

$L(e^{\lambda t}\alpha) = \lambda e^{\lambda t}\alpha - Ae^{\lambda t}\alpha = e^{\lambda t}(\lambda\alpha - A\alpha)$

$L(X) = 0$ iff $A\alpha = \lambda\alpha$, eigen problem

$(\lambda I_n - A)\alpha = 0$

non-trivial solution : $\det(\lambda I_n - A) = 0$

solutions : $X_j(t) = e^{\lambda_j t}\alpha_j$

$\tilde{X}(t) = Qe^{Dt}$, $Q = (\alpha_1\ \alpha_2 \cdots \alpha_n)$

$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

general case with multiplicity

$A \in M_n(\mathbb{C})$, n by n matrix over complex field.

let λ be eigenvalue, $\mathcal{E}(A, \lambda)$ eigenspace

$\therefore \mathcal{E}(A, \lambda) = N(A - \lambda I_n)$ nullspace

let $\mathcal{F}(A, \lambda) = N((A - \lambda I_n)^m)$, $m > 0$

$$\Downarrow$$

$$(A - \lambda I_n)^m \alpha = 0$$

From primary decomposition theorem,

if $\lambda_1, \cdots, \lambda_k$ are eigenvalues of A, with multiplicity

$m_1, \cdots, m_k$, $m_1 + \cdots + m_k = n$

then $\dim(\mathcal{F}(A, \lambda_j)) = m_j$

$$\mathbb{C}^n = \mathcal{F}(A, \lambda_1) \oplus \cdots \oplus \mathcal{F}(A, \lambda_k)$$

$$\xi = \xi_1 + \cdots + \xi_k, \quad \xi_j \in \mathcal{F}(A, \lambda_j)$$

$$x(t) = e^{At}\xi = e^{At}(\xi_1 + \cdots + \xi_k)$$

$$e^{At}\xi_j = e^{\lambda_j I_n t + (A - I_n \lambda_j)t}\xi_j = e^{\lambda_j t}\sum_{p=0}^{m_j - 1} \frac{t^p}{p!}(A - I_n \lambda)^p \xi_j = P_j(t) e^{\lambda_j t}$$

$$\therefore x(t) = e^{\lambda_1 t} P_1(t) + \cdots + e^{\lambda_k t} P_k(t)$$

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let α_{ij}, $i = 1, \dots, m_j$ be a basis for $F(A, \lambda_j)$

then α_{ij}, $i = 1, \dots, m_j$, $j = 1, \dots, k$ is a basis for $\mathbb{C}^n$. $X_{ij}(t) = e^{\lambda_j t} P_{ij}(t)$

proof: if $\xi = \alpha_{ij}$, $X' = AX$, $X_{ij}(0) = \alpha_{ij}$

suppose α_{ij} are dependent, $\sum c_{ij} \alpha_{ij} = 0$

let $\alpha_j = \sum_i c_{ij} \alpha_{ij}$

$\therefore \alpha_1 + \dots + \alpha_k = 0 \overset{\text{primary decomposition theorem}}{\Longrightarrow} \alpha_1, \dots, \alpha_k = 0$

$\Downarrow$ α_{ij}, $i = 1, \dots, m_j$ is a basis

$\alpha_{ij} = 0$

let A have n distinct eigenvalues, $\lambda_1 \dots \lambda_n$

$\tilde{X}(t) = Q e^{Dt}$ a basis

$X_A(t) = e^{At}$ a basis

$\therefore Q e^{Dt} C = e^{At} \overset{t=0}{\Longrightarrow} Q = C^{-1}$

$\therefore e^{At} = Q e^{Dt} Q^{-1}$

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Examples:

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} \quad p_A(\lambda) = (\lambda+2)^3, \quad \lambda = -2, \ m = 3.$$

$$\therefore X(t) = e^{-2t}(\alpha + \beta_1 t + \beta_2 t^2)$$

$$\beta_1 = (A + 2I_3)\alpha \quad \beta_2 = \frac{1}{2}(A + 2I_3)^2\alpha \quad (A+2I_3)^3\alpha = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\alpha = E_1 : \beta_1 = 0, \ \beta_2 = 0 \qquad \alpha = E_2 : \beta_1 = E_1, \ \beta_2 = 0$

$\alpha = E_3 : \beta_1 = E_2, \ \beta_2 = E_1/2$

$$\therefore \tilde{X}(t) = e^{-2t}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Real space

let A be Real, $A \in M_n(\mathbb{R})$

if $\lambda \in \mathbb{C}$ (complex number)

$\bar{\lambda}$ is also an eigenvalue, $\because \overline{p_A(\lambda)} = p_A(\bar{\lambda})$ ← same multiplicities

$\overline{(A - \lambda I_n)\alpha} = 0$ iff $(A - \bar{\lambda} I_n)\bar{\alpha} = 0$

⇓

Isomorphism

$x_i(t) = e^{\lambda t} p_i(t) \quad \bar{x}_i(t) = e^{\bar{\lambda} t}\bar{p}_i(t), \quad i = 1, \ldots, m$

Thus, $X_1, \cdots, X_m, \bar{X}_1, \cdots, \bar{X}_m$ independent solutions

$$\begin{array}{cccc} X_1 & X_m & \bar{X}_1 & \bar{X}_m \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \alpha_1 & \alpha_m & \bar{\alpha}_1 & \bar{\alpha}_m \end{array}$$

$\therefore \mathrm{Re}(X_j), \mathrm{Im}(X_j)$ is independent solution set

$\because$ let $X_k = a_k + i b_k$ $\qquad a_k = \dfrac{X_k + \bar{X}_k}{2}$

$\qquad b_k = \dfrac{X_k - \bar{X}_k}{2i}$

Higher-order equations

$$x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_0 x = 0, \quad a_j \in \mathbb{C}$$

let $\qquad y_1 = x$

$\qquad y_2 = x'$

$\qquad \vdots$

$\qquad y_n = x^{(n-1)}$

$\therefore y_n' = -a_0 y_1 - \cdots - a_{n-1} y_n$

$Y' = AY$

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & 1 & \\ 0 & & \ddots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix}$$

$$p_A(\lambda) = \det(\lambda I_n - A) = \det \begin{pmatrix} \lambda & -1 & & 0 \\ 0 & \lambda & -1 & \\ 0 & & \ddots & -1 \\ a_0 & a_1 & \cdots & a_{n-1}+\lambda \end{pmatrix}$$

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$$= \det \begin{pmatrix} \lambda & -1 & & \\ & \lambda & -1 & \\ & & \ddots & \\ 0 & a_1 + \frac{a_0}{\lambda} & & \end{pmatrix} = \det \begin{pmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \lambda & -1 & \\ & & & \ddots & \\ 0 & 0 & a_2 + \frac{1}{\lambda}\left(a_1 + \frac{a_0}{\lambda}\right) & & \end{pmatrix} \cdots$$

$$\therefore P_A(\lambda) = \det \begin{pmatrix} \lambda & -1 & & 0 \\ & \lambda & -1 & \\ & & \ddots & \\ 0 & & & -1 \\ 0 & 0 & 0 & q(\lambda) \end{pmatrix} \qquad q(\lambda) = \lambda + a_{n-1} + \frac{a_{n-2}}{\lambda} + \cdots + \frac{a_0}{\lambda^{n-1}}$$

$$= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$$

<u>solution basis :</u>

$$L(e^{\lambda t}) = p(\lambda) e^{\lambda t}$$

where $L(x) = x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_0 x$

note that L is linear

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$$

if $p(\lambda_1) = 0$, then $e^{\lambda_1 t}$ is a solution

if λ_1 is a root of multiplicity m_1

$$p(\lambda_1) = p'(\lambda_1) = \cdots = p^{(m_1-1)}(\lambda_1) = 0 \ , \ p^{(m_1)} \neq 0$$

$$\therefore p(\lambda) = (\lambda - \lambda_1)^{m_1} q(\lambda)$$

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$$\frac{\partial}{\partial \lambda^{\ell}} L(e^{\lambda t}) = L\left(\frac{\partial}{\partial \lambda^{\ell}} e^{\lambda t}\right) = L(t^{\ell} e^{\lambda t})$$

$$= \frac{\partial}{\partial \lambda^{\ell}}\left[p(\lambda) e^{\lambda t}\right]$$

$$= \left[p^{(\ell)}(\lambda) + \ell p^{(\ell-1)} t + \cdots + \binom{\ell}{j} p^{(\ell-j)} t^{j} + \cdots + p(\lambda) t^{\ell}\right] e^{\lambda t}$$

when $\ell = 0, 1, \ldots, m-1$, $t^{\ell} e^{\lambda_i t}$ is a solution

Jordan canonical form

let $\phi : V \to V$ n-dimensional vector space

$\lambda_1, \cdots, \lambda_r$ eigenvalues of ϕ

$$V = \ker(\phi - \lambda_1 I)^{s_1} \oplus \cdots \oplus \ker(\phi - \lambda_r I)^{s_r}$$

let $W_i := \ker(\phi - \lambda I)^i$, observe that $W_1 \subset W_2 \cdots \subset W_i \cdots$

$\because$ if $(\phi - \lambda I)x = 0$, $\Rightarrow (\phi - \lambda I)^2 x = (\phi - \lambda I)0 = 0$

since $\dim(V)$ is finite, $W_t = W_{t+1} = \cdots$

note that $\ker(\phi - \lambda I)^t \cap \mathrm{Im}(\phi - \lambda I)^t = 0$

$\because$ if $\exists x \in \ker(\phi - \lambda I)^t \cap \mathrm{Im}(\phi - \lambda I)^t$

let $x = (\phi - \lambda I)^t y$

$(\phi - \lambda I)^t x = (\phi - \lambda I)^{2t} y = 0$

since $W_t = W_{2t}$, $x = 0$ //

$\therefore \dim(V) = \dim(\ker(\phi - \lambda I)^t) + \dim(\mathrm{Im}(\phi - \lambda I)^t)$

$\therefore V = \ker(\phi - \lambda I)^t \oplus \mathrm{Im}(\phi - \lambda I)^t$

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note that both $\ker(\phi-\lambda I)^t$ and $\operatorname{Im}(\phi-\lambda I)^t$ are invariant subspaces

$$0 = \phi \cdot 0 = \phi(\phi-\lambda I)^t x = (\phi-\lambda I)^t \phi x \Rightarrow \phi x \in \ker(\phi-\lambda I)^t$$

$$\phi y = \phi(\phi-\lambda I)^t x = (\phi-\lambda I)^t \phi x \Rightarrow \phi y \in \operatorname{Im}(\phi-\lambda I)^t$$

put $\lambda = \lambda_1$,

$$V = \ker(\phi-\lambda_1 I)^t \oplus \operatorname{Im}(\phi-\lambda_1 I)^t$$

recursively, let $V' = \operatorname{Im}(\phi-\lambda_1 I)^t$, $\dim(V') < \dim(V)$

eigenvalues : $\lambda_2, \dots, \lambda_r$

~~$V = (\phi-\lambda_2 I)^{t'} \oplus$~~

$$V' = \ker(\phi-\lambda_2 I)^{t'} \oplus \operatorname{Im}(\phi-\lambda_2 I)^{t'}$$

inductively,

$$V' = \ker(\phi-\lambda_2 I)^{s_2} \oplus \cdots \oplus \ker(\phi-\lambda_r I)^{s_r}$$

$$\therefore V = \ker(\phi-\lambda_1 I)^{s_1} \oplus \cdots \oplus \ker(\phi-\lambda_r I)^{s_r}$$

[primary decomposition theorem]

note that each eigenspace is independent,

i.e. $\ker(\phi-\lambda_i I)^{s_i} \cap \ker(\phi-\lambda_j I)^{s_j} = 0$, $i \neq j$

$$\because (\phi-\lambda_i I)^{s_i} x = 0 \Rightarrow \sum \binom{s_i}{k} (-\lambda_i)^{s_i-k} \phi^k x$$

$$= (\lambda_j - \lambda_i)^{s_i} x = 0 \Rightarrow x = 0$$

Wildon's lemma:

let $T: V \to V$, $T^s = 0$, then $\exists\, u_1, \dots, u_k$
$a_1, \dots, a_k$

$T^{a_i}(u_i) = 0$

$V = u_1, T(u_1), \dots, T^{a_1 - 1}(u_1), \dots, u_k, T(u_k), \dots, T^{a_k - 1}(u_k)$

if $T: V \to 0$, then $u_1, \dots, u_k$ = basis of V,
$a_1, \dots, a_k = 1$

by induction proof:

$\dim(V) = 1$, $T^s = 0$ only if $T = 0$, i.e. $T: V \to 0$

$\dim(V) = n$, ~~xxx~~

$\dim(\mathrm{Im}(T))$ cannot be n, $\because$ this implies
T is one-to-one, $T^s \neq 0$

$\therefore$ $0 < \dim(\mathrm{Im}(T)) < n$

by induction hypothesis,
$T^{b_i}(v_i) = 0$, $i = 1, \dots, l$
$v_1, T(v_1), \dots, T^{b_1 - 1}(v_1), \dots, v_l, \dots, T^{b_l - 1}(v_l)$ are basis of $\mathrm{Im}(T)$

let $v_i = T(w_i)$

$\therefore T^{b_1-1}(v_1), T^{b_2-1}(v_2), \cdots$ are independent and $\in \ker(T)$

$\because T(T^{b_1-1}(v_1)) = 0$

by Steinitz theorem,

$T^{b_1-1}(v_1), \cdots, T^{b_\ell-1}(v_\ell), z_1, z_2, \cdots z_m$ basis of $\ker(T)$

$\Downarrow$

$w_1, T(w_1), \cdots, T^{b_1}(w_1), \cdots, w_\ell, \cdots, T^{b_\ell}(w_\ell), z_1, \cdots z_m$

are basis of V

note that $T^{b_1+1}(w_1) = \cdots = T^{b_\ell+1}(w_\ell) = 0$

by nullity T,

because we know that $T(w_1), T^2(w_1), \cdots, T^{b_\ell}(w_\ell)$

are independent, and furthermore $T^{b_1}(w_1), T^{b_2}(w_2), \cdots, z_m$

are independent.

$\underset{T^{b_1-1}(v_1)}{\shortparallel} \quad \underset{T^{b_2-1}(v_2)}{\shortparallel}$

$$\dim(V) = \dim(\mathrm{Im}(T)) + \dim(\ker(T))$$
$$= b_1 + \cdots + b_\ell + \ell + m$$
$$= (1+b_1) + \cdots + (1+b_\ell) + m \quad //$$

$T(x_i) = 0$

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an alternative way to prove $x, (\phi - \lambda I)x, (\phi - \lambda I)^2 x, \dots, (\phi - \lambda I)^{p-1} x$

independent

if dependent, $\sum_{j=0}^{k} b_j (\phi - \lambda I)^j x = 0$

$$0 = b_k (\phi - \lambda I)^{k+p-k} x = -\sum_{j=0}^{k-1} b_j (\phi - \lambda I)^{j+p-k} x$$

$$\sum_{j=p-k}^{p-1} b_{j-p+k} (\phi - \lambda I)^j x = 0$$

$$\Downarrow$$

$$0 = b_{k-1} (\phi - \lambda I)^p x = -\sum_{j=p-k}^{p-2} b_{j-p+k} (\phi - \lambda I)^{j+1} x$$

$$\sum_{j=p-k+1}^{p-1} b_{j-p+k-1} (\phi - \lambda I)^j x$$

$$\Downarrow \dots$$

$$c (\phi - \lambda I)^{p-1} x = d (\phi - \lambda I)^{p-2} x$$

$$\therefore (\phi - \lambda I) x = \frac{d}{c} x$$

$\therefore \lambda + \frac{d}{c}$ is a new eigenvalue, contradiction.

Proof of Jordan normal form

since $V = \ker(\phi - \lambda_1 I)^{s_1} \oplus \dots \oplus \ker(\phi - \lambda_r I)^{s_r}$

$$\therefore \phi = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_r \end{pmatrix}$$

J_i is block matrix

$J_i = \ker(\phi - \lambda_i I)^{s_i}$

Each block matrix J_i can be triangularized by Schur's theorem,

$$K(\lambda_i) = \begin{pmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{pmatrix} = \lambda_i I + N$$

N ← nilpotent

$$N^{s_i - 1} \neq 0, \quad N^{s_i} = 0$$

by Wilson's lemma,

$$y_k = N^{k-1} x, \quad k = 1, \dots, s_i$$

independent, basis of ~~$\ker(\phi - \lambda_i I)^{s_i}$~~ V^{s_i}

let $Q = \begin{pmatrix} y_m & \cdots & y_1 \end{pmatrix}$ $\quad Q^{-1} = \begin{pmatrix} y_m \\ \vdots \\ y_1 \end{pmatrix}$

$$NQ = \begin{pmatrix} 0 & y_m & \cdots & y_2 \end{pmatrix}$$

$$Q^{-1} N Q = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$$

$$\therefore Q^{-1} K(\lambda_i) Q = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

$$\therefore \phi = \Omega^{-1} \begin{pmatrix} J_{\lambda_1, 1} & & & \\ & J_{\lambda_2, 1} & & \\ & & \ddots & \\ & & & J_{\lambda_r, 1} \end{pmatrix} \Omega$$

$\lambda_1, \dots, \lambda_r$ need not be distinct

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Cayley - Hamilton theorem

$p(\lambda) = \det(\lambda I_n - A)$

$p(A) = O_n$

$\cancel{p(\lambda) = (x}$

$p(\lambda) = (\lambda - \lambda_1)^{s_1} \cdots (\lambda - \lambda_r)^{s_r}$

$p(A) = (A - \lambda_1 I)^{s_1} \cdots (A - \lambda_r I)^{s_r}$

$$\downarrow$$

$$\begin{pmatrix} \begin{smallmatrix} 0 & 1 & \\ & 0 & \ddots \end{smallmatrix} & & & \\ & J_{2,1} & & \\ & & \ddots & \\ & & & J_{r,1} \end{pmatrix} \xrightarrow{\text{self multiply } s_1 \text{ times}} \begin{pmatrix} 0 & & & \\ & * & & 0 \\ & & * & \\ 0 & & & \ddots\, * \end{pmatrix}$$

$$\Downarrow$$

$p(A) = O_n$